

Vacuum $f(R)$ thick brane solution with a Gaussian warp function

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Abstract

This work deals with $f(R)$ modified gravity in five dimensional space-time. The Gaussian thick brane is shown to be an exact solution in the frame work of $f(R)$ gravity in five dimensions with a bulk cosmological constant. Response of the brane to gravitational fluctuations and concordance with the Starobinsky model is addressed. It is shown that the matter which supports the Starobinsky $f(R)$ solution with the background geometry being flat FLRW with a Gaussian warp function, behaves like a radiation dominated era of universe, gradually changing to a dark energy dominated era.

1 Introduction

In the past two decades, the brane world concept has been the subject of intensive investigations in connection with the recent developments in superstring/M-theories[1]. The investigations were first initiated in the work by Kaluza and Klein in 1920s in order to unify two fundamental forces electromagnetism and gravitation within the framework of a unified five-dimensional theory. In this model, the size of the extra dimensions are compacted to the Planck scale[2]. However, in the brane world model, the sizes of the extra dimensions are about a few TeV^{-1} [3]-[5], millimeters [6] or very large [7, 8].

According to the brane world model, the standard-model particles are confined to a hypersurface, called a brane, immersed in a higher-dimensional spacetime called the bulk. It is postulated that the matter fields are in the brane while the gravitational waves are free to propagate into the bulk. The success of extra dimensions has brought a solution for a number of insoluble problems in high-energy physics: the problem of mass hierarchy, stability, etc. This idea is carried by many theories, but the main ones in this context are the one proposed by Arkani-Hamed, Dimopoulos and Dvali (ADD) [6], [9, 10] and the so called, Randall-Sundrum (RS) model [11, 12]. Particularly, the RS model has been advocated as a simple one.

In fact there are two RS models within the same framework. In the RS-I model, the extra dimension appears due to the anti-de-Sitter (AdS) geometry along the fifth dimension. On the other hand, this model deals with two $D3$ branes on the S_1/Z_2 orbifold along the extra dimension [13]. The presence of two singular $D3$ branes with opposite tensions is needed for this model. It should be emphasized that, with this model the hierarchy problem can be solved without restoring to the large compactified volume of the extra-dimension as proposed by ADD.

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In the RS-II model, the authors considered a 3-brane (the four-dimensional Minkowski spacetime) with a positive tension embedded in a five-dimensional anti de Sitter (AdS_5) spacetime. They showed that there exists a massless graviton (zero mode) and massive gravitons (Kaluza-Klein modes). The Newtonian gravity on the 3-brane is reproduced by the massless graviton. Therefore, by intuition, the massive modes which are the effect of the existence of the extra dimensions, cause a correction to the Newtonian gravity. They also showed that in the low energy limit the Newtonian gravity can be recovered [14], [15]. Since there is only one $D3$ brane, this model can not address the hierarchy problem[13].

The brane should have some thickness which yields new possibilities and new problems[16]. This kind of brane should fulfill two main requirements. One is that the solutions should be regular and asymptotically flat, or de Sitter, the other is that the matter should be restricted close to the brane. In the RS-II model, the 3-brane has no thickness, and the geometry has a singularity at the brane location. In order to escape the singularity, the extension of RS-II model by replacing the 3-brane by a smooth thick brane obtained from a background scalar field can be invoked[17]-[19]. With this configuration of the thick brane, the bulk is not an AdS_5 spacetime[20]. Another way for generating thick branes instead of using scalar fields is to build them from pure geometry [21]-[24]. In these papers, the gravitational zero mode as well as the decoupling of the massive Kaluza-Klein modes are investigated.

In this paper, we use pure geometry for generating thick branes by invoking $f(R)$ theory where the gravitational Lagrangian is a function of Ricci scalar. The $f(R)$ theory was first created for studying the evolution of the universe[25]-[27]. In the works [28, 29], the authors consider thick RS-II brane world solutions in pure $f(R)$ gravity. In [28], numerical solutions obtained. Also an analytical thick brane solution is given in [29].

In this work, we derive a pure $f(R)$ solution by supposing a Gaussian thick brane. The model will be presented in the next section. In section 3, the gravitational fluctuations and the localization of gravity in the vicinity of the brane are discussed. In section 4, the $f(R)$ gravity in the Einstein frame is investigated and the corresponding scale factor and scalar potential versus the scalar curvature are derived. In section 5, we consider the Starobinsky $f(R)$ model with the background geometry being flat FLRW universe with a Gaussian warp function and we study the behavior of the matter which supports the solution.

2 The Model and the $f(R)$ Solution

We begin with considering a pure $f(R)$ Lagrangian which is an analytic function of Ricci scalar in a five dimensional spacetime. The action specifying the dynamics of the brane-bulk system without matter source is[30]

$$S = \frac{1}{2\kappa_5^2} \int d^4x dy \sqrt{-g} f(R), \quad (1)$$

we use $\kappa_5^2 = \frac{8\pi}{M_*^3}$, where M_* is the five dimensional Planck scale, y denotes the extra dimension and g is the determinant of the five dimensional metric. We consider the flat and static brane embedded in a five dimensional bulk which has the following metric

$$ds^2 = e^{2A(y)} \eta_{\alpha\beta} dx^\alpha dx^\beta + dy^2, \quad (2)$$

where $e^{2A(y)}$ is the warp function and $\eta_{\alpha\beta}$ is the four dimensional Minkowski metric with signature $(-, +, +, +)$. Throughout this paper, Greek letters α, β run over 0, 1, 2, 3 and capital Latin ones $A, B = 0, 1, 2, 3, 4$ are used to represent the brane and bulk indices, respectively. In the present

case, we choose the warp function $e^{2A(y)} = e^{-\lambda y^2}$ which has a Gaussian shape with Z_2 symmetry and $\sqrt{\lambda}$ is the inverse of the brane thickness $\Delta = \frac{1}{\sqrt{\lambda}}$.

To obtain the equations of motion, one can vary the action (1) in the usual manner which gives

$$R_{AB}F(R) - \frac{1}{2}g_{AB}f(R) + (g_{AB}\square^{(5)} - \nabla_A\nabla_B)F(R) = 0, \quad (3)$$

where $\square^{(5)} = g^{AB}\nabla_A\nabla_B$ is the five-dimensional d'Alembert operator and $F(R) \equiv \frac{df(R)}{dR}$. By inserting metric (2) in (3), the following field equations in the absence of matter can be obtained

$$f(R) + 2\lambda(4\lambda y^2 - 1)F(R) + 6\lambda y\dot{F}(R) - 2\ddot{F}(R) = 0, \quad (4)$$

and

$$-8\lambda(\lambda y^2 - 1)F(R) - 8\lambda y\dot{F}(R) - f(R) = 0, \quad (5)$$

where dot stands for the derivative with respect to y . Adding the above equations one can obtain

$$\ddot{F} + \lambda y\dot{F} - 3\lambda F = 0, \quad (6)$$

which is a second order differential equation for $F(R(y))$. Note that the Ricci scalar for metric (2) and the mentioned Gaussian warp function is given by

$$R = -4\lambda(5\lambda y^2 - 2), \quad (7)$$

which gives

$$y = \pm \frac{1}{10\lambda}\sqrt{40\lambda - 5R} \quad (8)$$

Hence, by solving Eq. (5), the function $F(R(y))$ can be explicitly obtained given by

$$F(R(y)) = c_1 y(3 + \lambda y^2) + c_2 e^{-\lambda y^2/2} \text{hypergeom}\left([2], [\frac{1}{2}], \frac{1}{2}\lambda y^2\right). \quad (9)$$

Substituting $F(R(y))$ into Eq. (4), $f(R(y))$ and consequently $f(R)$ can be calculated and is given by

$$\begin{aligned} f(R) = & C(8\lambda - R)^{3/2} \left[1 + \frac{1}{100\lambda}(8\lambda - R) \right] + 8c_2 \lambda e^{\frac{R-8\lambda}{40\lambda}} \text{hypergeom}\left([2], [\frac{1}{2}], \frac{8\lambda - R}{40\lambda}\right) \\ & - \frac{8}{5}c_2(8\lambda - R)e^{\frac{R-8\lambda}{40\lambda}} \text{hypergeom}\left([3], [\frac{3}{2}], \frac{8\lambda - R}{40\lambda}\right), \end{aligned} \quad (10)$$

in which $C \equiv \pm c_1 \frac{\sqrt{5}}{5\lambda}$, and the “ \pm ” signs account for two possible branches of solutions ($y = \pm \frac{1}{10\lambda}\sqrt{40\lambda - 5R}$). Here we shall assume $c_2 = 0$, which leads to

$$f(R) = C(8\lambda - R)^{3/2} \left[1 + \frac{1}{100\lambda}(8\lambda - R) \right]. \quad (11)$$

The expansion of $f(R)$ around $R = 0$ up to the third order is

$$f(R) = \frac{432}{25}\sqrt{2}C\lambda^{\frac{3}{2}} - \frac{17}{5}\sqrt{2\lambda}CR + \frac{21C}{80\sqrt{2\lambda}}R^2, \quad (12)$$

where for small curvature that is $R \rightarrow 0$, the $f(R)$ function goes to a constant value

$$\lim_{R \rightarrow 0} f(R) = \frac{432}{25} \sqrt{2} C \lambda^{\frac{3}{2}}. \quad (13)$$

Note that Eq. (11) sets a maximum curvature

$$R_{max} = 8\lambda. \quad (14)$$

Inserting Eq. (12) into the action (1) and comparing it with the Einstein-Hilbert action with a cosmological constant Λ [31], i.e.

$$S_{EH} = \frac{1}{2\kappa_5^2} \int d^4x dy \sqrt{-g} (R^{(5)} - 2\Lambda_5), \quad (15)$$

leads to the following constraints

$$\sqrt{2\lambda}C = -\frac{5}{17}, \quad \Lambda_5 = \frac{216}{85}\lambda \quad \text{and} \quad c_1 = \pm \frac{25\sqrt{\lambda}}{17\sqrt{10}}. \quad (16)$$

3 Gravitational Fluctuations

In this section, we shall consider the gravitational fluctuations of the metric (2), following the usual formalism[32].

$$ds^2 = e^{2A(y)} (\eta_{\alpha\beta} + h_{\alpha\beta}) dx^\alpha dx^\beta + dy^2, \quad (17)$$

where $h_{\alpha\beta} = h_{\alpha\beta}(x^\rho, y)$ is a fluctuation which depends on all coordinates. By defining $a(y) \equiv e^{A(y)}$, the following fluctuations for the Riemann tensor and the Ricci scalar are obtained

$$\begin{aligned} \delta R_{\alpha\beta} &= -\frac{1}{2}(\Box^{(4)} h_{\alpha\beta} + \partial_\alpha \partial_\beta h - \partial_\beta \partial_\sigma h_\alpha^\sigma - \partial_\alpha \partial_\sigma h_\beta^\sigma) - 2aa'h'_{\alpha\beta} \\ &\quad - 3h_{\alpha\beta}a'^2 - ah_{\alpha\beta}a'' - \frac{a^2 h''_{\alpha\beta}}{2} - \frac{a\eta_{\alpha\beta}a'h'}{2}, \\ \delta R_{\alpha 5} &= \frac{1}{2}\partial_y(\partial_\lambda h_\alpha^\lambda - \partial_\alpha h), \quad \delta R_{55} = -\frac{1}{2}\left(\frac{2a'h'}{a} + h''\right), \\ \delta R &= \delta(g^{\alpha\beta} R_{\alpha\beta}) = -\frac{\Box^{(4)} h}{a^2} + \frac{\partial_\alpha \partial_\beta h^{\alpha\beta}}{a^2} - \frac{a'}{a}5h' - h'', \end{aligned} \quad (18)$$

where $\Box^{(4)} = \eta^{\alpha\beta} \partial_\alpha \partial_\beta$, is d'Alembert operator in four-dimensions, prime denotes derivative with respect to y and $h = \eta^{\alpha\beta} h_{\alpha\beta}$ is the trace of the tensor perturbations.

In order to simplify the perturbed equations, we use the transverse-traceless gauge given by

$$h = 0 = \partial_\mu h^\mu{}_\nu. \quad (19)$$

With this choice, only $\delta R_{\mu\nu}$ will not vanish. The perturbation along the $f(R)$ equations of motion (3) reads

$$\begin{aligned} &\delta R_{AB}F(R) + R_{AB}F(R)_{,R}\delta R - \frac{1}{2}\delta g_{AB}f(R) - \frac{1}{2}g_{AB}F(R)\delta R \\ &+ \delta(g_{AB}\Box^{(5)}F(R)) - \delta(\nabla_A \nabla_B F(R)) = 0. \end{aligned} \quad (20)$$

For the above equations, we have

$$\begin{aligned}\nabla_A \nabla_B F(R) &= (\partial_A \partial_B - \Gamma_{AB}^P \partial_P) F(R), \\ g_{AB} \square^{(5)} F(R) &= g_{AB} g^{MN} (\nabla_M \nabla_N F(R)),\end{aligned}\quad (21)$$

and also

$$\delta(\nabla_A \nabla_B F(R)) = (\partial_A \partial_B - \Gamma_{AB}^P \partial_P)(F(R),_R \delta R) - \delta \Gamma_{AB}^P \partial_P F(R), \quad (22)$$

$$\begin{aligned}\delta(g_{AB} \square^{(5)} F(R)) &= \delta g_{AB} \square^{(5)} F(R) + g_{AB} \delta g^{MN} (\nabla_M \nabla_N F(R)) \\ &+ g_{AB} g^{MN} \delta(\nabla_M \nabla_N F(R)).\end{aligned}\quad (23)$$

By using the transverse and traceless gauge, which leads to $\delta R = 0$, the above equations will become

$$\begin{aligned}\delta(\nabla_A \nabla_B F(R)) &= -\delta_A^\rho \delta_B^\sigma \delta \Gamma_{\rho\sigma}^5 F'(R), \\ \delta(g_{AB} \square^{(5)} F(R)) &= \delta_A^\rho \delta_B^\sigma \delta g_{\rho\sigma} \square^{(5)} F(R).\end{aligned}\quad (24)$$

Therefore, in this gauge, we have

$$\begin{aligned}&\delta(g_{AB} \square^{(5)} F(R)) - \delta(\nabla_A \nabla_B F(R)) \\ &= \delta_A^\rho \delta_B^\sigma a^2 \left[h_{\rho\sigma} \left(3 \frac{a'}{a} F'(R) + F''(R) \right) - \frac{1}{2} F'(R) h'_{\rho\sigma} \right].\end{aligned}\quad (25)$$

The perturbed $f(R)$ equations of motion (20) reduce to

$$\begin{aligned}&\delta R_{AB} F(R) - \frac{1}{2} \delta g_{AB} f(R) \\ &+ \delta_A^\rho \delta_B^\sigma a^2 \left[h_{\rho\sigma} \left(3 \frac{a'}{a} F'(R) + F''(R) \right) - \frac{1}{2} h'_{\rho\sigma} F'(R) \right] = 0.\end{aligned}\quad (26)$$

By plugging (18) into (26), we obtain the (α, β) components of the perturbed $f(R)$ equations as

$$\begin{aligned}&\left(-\frac{1}{2} \square^{(4)} h_{\alpha\beta} - 3 h_{\alpha\beta} a'^2 - 2 a a' h'_{\alpha\beta} - a h_{\alpha\beta} a'' - \frac{a^2 h''_{\alpha\beta}}{2} \right) F(R) \\ &- \frac{1}{2} a^2 h_{\alpha\beta} f(R) + a^2 \left[h_{\alpha\beta} \left(3 \frac{a'}{a} F'(R) + F''(R) \right) - \frac{1}{2} h'_{\alpha\beta} F'(R) \right] = 0.\end{aligned}\quad (27)$$

On the other hand, the (α, α) components of $f(R)$ equations (3) is

$$f(R) + 2F(R) \left[3 \left(\frac{a'}{a} \right)^2 + \frac{a''}{a} \right] - 6F'(R) \frac{a'}{a} - 2F''(R) = 0. \quad (28)$$

By simplifying Eq. (27), one can get

$$\left(-\frac{1}{2} \square^{(4)} h_{\alpha\beta} - 2 a a' h'_{\alpha\beta} - \frac{a^2 h''_{\alpha\beta}}{2} \right) F(R) - \frac{1}{2} a^2 h'_{\alpha\beta} F'(R) = 0. \quad (29)$$

Consequently, the (α, β) components of the perturbed $f(R)$ equations read

$$\left(a^{-2} \square^{(4)} h_{\alpha\beta} + 4 \frac{a'}{a} h'_{\alpha\beta} + h''_{\alpha\beta} \right) F(R) + h'_{\alpha\beta} F'(R) = 0. \quad (30)$$

This can be written as

$$\square^{(5)} h_{\alpha\beta} = \frac{F'(R)}{F(R)} \partial_y h_{\alpha\beta}. \quad (31)$$

Introducing a coordinate transformation

$$dz = a^{-1} dy, \quad (32)$$

the perturbed equation (31) can be written as

$$\left[\partial_z^2 + \left(3 \frac{\partial_z a}{a} + \frac{\partial_z F(R)}{F(R)} \right) \partial_z + \square^{(4)} \right] h_{\alpha\beta} = 0. \quad (33)$$

Following [32], we look for solutions of the form $h_{\alpha\beta}(x^\rho, z) = (a^{-3/2} F(R)^{-1/2}) \epsilon_{\alpha\beta}(x^\rho) \psi(z)$, in which $\epsilon_{\alpha\beta}(x^\rho)$ satisfies the transverse and traceless condition $\eta^{\alpha\beta} \epsilon_{\alpha\beta} = 0$ and $\partial_\alpha \epsilon_\beta^\alpha = 0$. Then the we end up with a Schrödinger like equation for $\psi(z)$

$$[\partial_z^2 - U(z)] \psi(z) = -m^2 \psi(z), \quad (34)$$

where the potential $U(z)$ is given by

$$U(z) = \frac{3}{4} \frac{a'^2}{a^2} + \frac{3}{2} \frac{a''}{a} + \frac{3}{2} \frac{a' F'(R)}{a F(R)} - \frac{1}{4} \frac{F'(R)^2}{F(R)^2} + \frac{1}{2} \frac{F''(R)}{F(R)}. \quad (35)$$

In order to understand the behavior of the potential $U(z)$, we use the coordinate transformation (32), and then obtain the potential as a function of the coordinate y . In Figure 1, we show the potential $U(y)$ for different values of c_1 and c_2 which are used in equation (9). The minimum of the potential is related to the stability of the solution. For the case with $c_1 = 0$ and $c_2 = 0.5$, there are two stable points. With $c_1 = 0.55$ and $c_2 = 0$, the potential is singular at $y = 0$. For $c_1 = 0.7, c_2 = 1.5$ and $c_1 = 0.45, c_2 = -1$, the potential has only one stable point.

One can also factorize the Schrödinger like equation (34) as

$$\left[\left(\partial_z + \left(\frac{3}{2} \frac{\partial_z a}{a} + \frac{1}{2} \frac{\partial_z F(R)}{F(R)} \right) \right) \left(\partial_z - \left(\frac{3}{2} \frac{\partial_z a}{a} + \frac{1}{2} \frac{\partial_z F(R)}{F(R)} \right) \right) \right] \psi(z) = -m^2 \psi(z), \quad (36)$$

which shows that there is no gravitational mode with $m^2 < 0$. As a result any solution of Eqs. (1), (2) is stable under the tensor perturbations. If the zero mode exists, it will have the form

$$\psi^{(0)}(z) = N_0 a^{3/2}(z) \sqrt{F(z)}, \quad (37)$$

with N_0 the normalization constant. A normalizable $\psi^{(0)}(z)$ leads to the Newton's law in four-dimensional gravity [8], [33]. The zero mode $\psi^{(0)}(z)$ is normalizable if

$$1 = \int_{-\infty}^{+\infty} |\psi^{(0)}(z)|^2 dz = N_0^2 \int_{-\infty}^{+\infty} e^{3A(z)} F(R(z)) dz = N_0^2 \int_{-\infty}^{+\infty} e^{2A(y)} F(R(y)) dy, \quad (38)$$

can be satisfied, which for our case it can not be integrated analytically. However, the numerical integration with the use of Eq. (9) gives the value $N_0 = 0.5541$, which confirms that for our solution, the gravitational zero mode is normalizable and can be localized on the brane. In this case, the Newton's law can be retrieved on the brane.

In general relativity, Eq. (31) will reduce to the five-dimensional Klein-Gordon equation for the massless spin-2 gravitons. Nevertheless, by having an arbitrary function $f(R)$ and non-constant curvature R , the equation for $h_{\alpha\beta}$ is completely different from the massless Klein-Gordon equation. Moreover, by using the transverse and traceless gauge, the perturbed equation always remains second order. Some of the application of these results are given in [34].

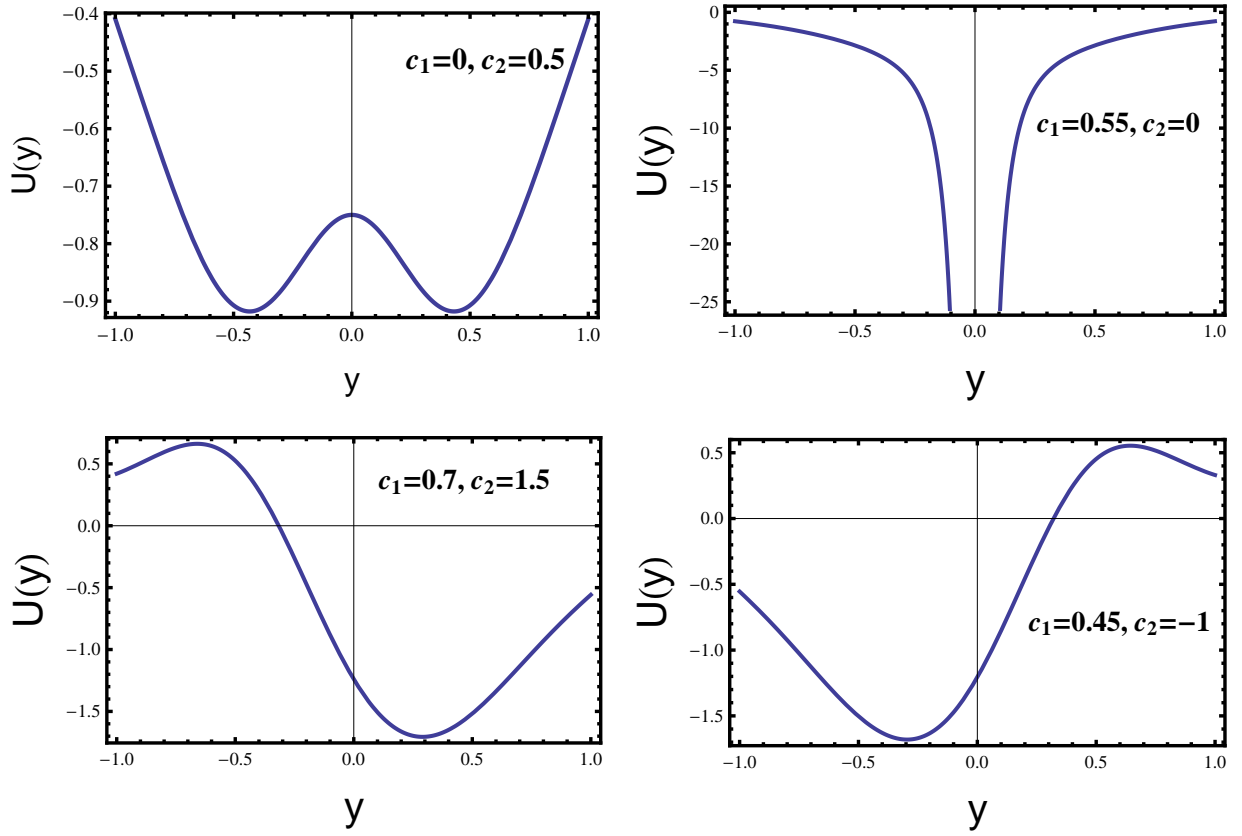


Figure 1: The plots depict the behavior of the potential $U(y)$ with respect to the coordinate y . In all figures it is assumed that the brane thickness $\Delta = 1$.

4 Einstein frame

In this section, in order to improve our understanding about the dynamics of the $f(R)$ model we transform to the Einstein frame. Practically, it is not easy to define the scalar degree of freedom for a $f(R)$ gravity. Instead, using the effective potential in a particular form will be convenient [35]. We use the following conformal transformation for switching to the Einstein frame [27], [36], [38]

$$\tilde{g}_{AB} = \Omega^2 g_{AB}, \quad (39)$$

where Ω^2 is the conformal factor and a tilde represents a quantity in Einstein frame. Under the conformal translation, the Ricci scalar transforms as [38]

$$R = \Omega^2 \tilde{R} + 8\tilde{g}^{AB}\Omega(\tilde{\nabla}_A\tilde{\nabla}_B\Omega) - 20\tilde{g}^{AB}\tilde{\nabla}_A\Omega\tilde{\nabla}_B\Omega, \quad (40)$$

where $\tilde{\nabla}_A$ is the covariant derivative in the conformal frame.

We can therefore rewrite the action (1) in the form

$$S = \int d^4x dy \sqrt{-\tilde{g}} \left(\frac{F(R)}{2\kappa_5^2} \Omega^{-5} R - \Omega^{-5} U \right), \quad (41)$$

where

$$U = \frac{F(R)R - f(R)}{2\kappa_5^2}. \quad (42)$$

In the above equations we used $\sqrt{-g} = \Omega^{-5}\sqrt{-\tilde{g}}$. Substituting Eq. (40) into (41) and making the choice $F(R) = \Omega^3$, then the action will reduce to

$$S = \int d^4x dy \sqrt{-\tilde{g}} \left(\frac{1}{2\kappa_5^2} [\tilde{R} - 12\Omega^{-2}\tilde{g}^{AB}\tilde{\nabla}_A\Omega\tilde{\nabla}_B\Omega] - \Omega^{-5} U \right). \quad (43)$$

By defining

$$\phi = 2\sqrt{\frac{3}{\kappa_5^2}} \ln \Omega, \quad V(\phi) = \Omega^{-5} U, \quad (44)$$

the action is simplified as

$$S = \int d^4x dy \sqrt{-\tilde{g}} \left(\frac{1}{2\kappa_5^2} \tilde{R} - \frac{1}{2} \tilde{g}^{AB} \tilde{\nabla}_A \phi \tilde{\nabla}_B \phi - V(\phi) \right). \quad (45)$$

For our model the conformal factor is

$$\Omega^2 = F(R)^{2/3} = \exp\left(\frac{\kappa\phi}{\sqrt{3}}\right). \quad (46)$$

Using the Eq. (44), the scalar field corresponding to our model (11) can be expressed as

$$\phi = \frac{2}{3} \sqrt{\frac{3}{\kappa_5^2}} \ln \left(\frac{C(68\lambda - R)\sqrt{8\lambda - R}}{40\lambda} \right). \quad (47)$$

Solving Eq. (47) to obtain R versus ϕ gives

$$R = 2 \left(24\lambda + 10^{4/3} \frac{C^2 \lambda^2}{\Phi(\phi)} + 10^{2/3} \frac{\Phi(\phi)}{C^2} \right), \quad (48)$$

where A is defined as $A \equiv \frac{2}{\sqrt{3}\kappa_5}$ and $\Phi(\phi)$ is given by

$$\Phi(\phi) \equiv \left[-C^4 \lambda^2 e^{2\phi/A} - 10C^6 \lambda^3 + \sqrt{\lambda^4 C^8 e^{2\phi/A} (e^{2\phi/A} + 20\lambda C^2)} \right]^{1/3}. \quad (49)$$

Using this expression for R , the $f(R(\phi))$ and consequently the field potential (44) takes the form

$$\begin{aligned} V(\phi) = & \frac{e^{-2\phi/3A}}{\kappa_5^2} \left(24\lambda + 10^{4/3} \frac{\Phi(\phi)}{C^2} + 10^{2/3} \frac{C^2 \lambda^2}{\Phi(\phi)} \right) \\ & - \frac{e^{-5\phi/3A}}{\kappa_5^2} \sqrt{2} C \left(-20\lambda - 10^{4/3} \frac{\Phi(\phi)}{C^2} - 10^{2/3} \frac{C^2 \lambda^2}{\Phi(\phi)} \right)^{3/2} \times \\ & \left(1 - \frac{20C^2 \lambda + \frac{10^{4/3} C^4 \lambda^2}{\Phi(\phi)} + 10^{2/3} \Phi(\phi)}{50C^2} \right). \end{aligned} \quad (50)$$

5 Flat FLRW Brane with Starobinsky $f(R)$ model

In this section, we consider a flat FLRW brane with a scale factor $a(t)$ depending on the cosmological time t

$$ds^2 = e^{-\lambda y^2} [-dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2)] + dy^2, \quad (51)$$

the Ricci scalar for metric (51) is given by

$$R = \frac{2}{a^2 e^{-\lambda y^2}} \left[3a\ddot{a} + 3\dot{a}^2 - 10a^2 \lambda^2 y^2 e^{-\lambda y^2} + 4a^2 \lambda e^{-\lambda y^2} \right], \quad (52)$$

where dot represents the time derivative. As it is seen, the Ricci scalar is a function of t and y . Here, we restrict ourselves to cases where the Ricci scalar has only y dependence. Hence, by putting the time dependent part of the Ricci scalar equals zero, that is $a(t)\ddot{a}(t) + \dot{a}(t)^2 = 0$, we will find $a(t) = \pm\sqrt{2c_1 t + 2c_2}$ which has a radiation like behavior, c_1 and c_2 being constants of integration. We put $c_2 = 0$ in order to have the same reference time for the big bang as in FLRW models. Note that, in our model as the scale factor goes to zero $a(t) \rightarrow 0$, the Ricci scalar does not diverge which is completely different with the FLRW cosmological model, where in the latter the scalar curvature will diverge.

Now with the above choice, the metric (51) reduces to

$$ds^2 = e^{-\lambda y^2} [-dt^2 + 2c_1 t(dr^2 + r^2 d\Omega^2)] + dy^2. \quad (53)$$

Similar to the calculations of section (1), we begin with the action (1) plus a matter term S_M , therefore the total action for $f(R)$ gravity takes the form

$$S = \frac{1}{2\kappa_5^2} \int d^4 x dy \sqrt{-g} f(R) + S_M(g_{AB}). \quad (54)$$

Variation with respect to the metric g_{AB} gives the $f(R)$ modified gravity as

$$F(R)R_{AB} - \frac{1}{2}f(R)g_{AB} - \nabla_A \nabla_B F(R) + g_{AB} \square F(R) = \kappa_5^2 T_{AB} \quad (55)$$

where $\square \equiv \nabla^C \nabla_C$ is the d'Alembert operator and

$$T_{AB} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{AB}}. \quad (56)$$

In order to clarify explicitly the behavior of the above field equations, we consider a perfect fluid which is characterized by

$$T_{AB} = (\rho + P)u_A u_B + P g_{AB}, \quad (57)$$

where ρ is the energy density, P is the pressure and u_A is the velocity vector. Moreover, we assume that the $f(R)$ function is the famous Starobinsky $f(R)$ model given by

$$f(R) = R + \alpha R^2, \quad (58)$$

which conforms with the expansion of our model up to R^2 and with $\Lambda = 0$ in the present case. Obviously $F(R) = 1 + 2\alpha R$ and the Ricci scalar $R = -4\lambda(5\lambda y^2 - 2)$. As a consequence, $f(R)$ and $F(R)$ will be functions of y .

We set up the field equations (55) which have the following components

$$F(R)R_{tt} - \frac{1}{2}f(R)g_{tt} + g_{tt}\square F = \kappa_5^2 T_{tt} = \kappa_5^2 \rho e^{-\lambda y^2} \quad (59)$$

$$F(R)R_{rr} - \frac{1}{2}f(R)g_{rr} + g_{rr}\square F = \kappa_5^2 T_{rr} = \kappa_5^2 T_{\theta\theta} = \kappa_5^2 T_{\phi\phi} = \kappa_5^2 P_r e^{-\lambda y^2} \quad (60)$$

and

$$F(R)R_{yy} - \frac{1}{2}f(R)g_{yy} - \nabla_y \nabla_y F(R) + \square F = \kappa_5^2 T_{yy} = \kappa_5^2 P_y. \quad (61)$$

The above equations reduce to

$$4F(R)H^2 A(t, y) + \frac{1}{2}f(R) - \square F(R) = \kappa_5^2 \rho(t, y), \quad (62)$$

$$4F(R)H^2 B(t, y) - \frac{1}{2}f(R) + \square F(R) = \kappa_5^2 P(t, y), \quad (63)$$

and

$$-40\alpha\lambda^4 y^4 - 64\alpha\lambda^3 y^2 + 6\lambda^2 y^2 + 32\alpha\lambda^2 = \kappa_5^2 P_y(y), \quad (64)$$

where $A(t, y)$ and $B(t, y)$ are given by

$$A(t, y) = \left[\frac{3}{4}e^{\lambda y^2} + \lambda(4\lambda y^2 - 1)t^2 \right], \quad (65)$$

$$B(t, y) = \left[\frac{1}{4}e^{\lambda y^2} - \lambda(4\lambda y^2 - 1)t^2 \right], \quad (66)$$

and H is the Hubble parameter defined as $H = \dot{a}(t)/a(t) = \frac{1}{2t}$. Inserting $f(R)$ and $F(R)$ as a function of y in Eqs. (62), (63) we obtain

$$\begin{aligned} \kappa_5^2 \rho(t, y) = & 4H^2 (1 - 8\alpha\lambda(5\lambda y^2 - 2)) \left(\frac{3}{4}e^{\lambda y^2} + \lambda(4\lambda y^2 - 1)t^2 \right) \\ & + (5\lambda y^2 - 2) (-2\lambda + 8\lambda^2\alpha(5\lambda y^2 - 2)) + 80\alpha\lambda^2(1 + 4\lambda y^2), \end{aligned} \quad (67)$$

$$\begin{aligned} \kappa_5^2 P(t, y) = & 4H^2 (1 - 8\alpha\lambda(5\lambda y^2 - 2)) \left(\frac{1}{4}e^{\lambda y^2} - \lambda(4\lambda y^2 - 1)t^2 \right) \\ & + (5\lambda y^2 - 2) (2\lambda - 8\lambda^2\alpha(5\lambda y^2 - 2)) - 80\alpha\lambda^2(1 + 4\lambda y^2). \end{aligned} \quad (68)$$

As it is seen from (64), the pressure in the y direction is only a function of y and independent of the cosmic time t .

An effective equation of state (EoS) parameter can be introduced as

$$w(t, y) = \frac{P(t, y)}{\rho(y, t)}, \quad (69)$$

which can be regarded as the dynamical EoS parameter of the model.

Furthermore, the energy conservation law $\nabla_A T^{A0} = 0$ gives

$$\dot{\rho} + \frac{3}{2t}P + \frac{3}{2t}\rho = 0, \quad (70)$$

which conforms with the FLRW continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0, \quad (71)$$

during radiation-dominated era where $H(t) = \frac{1}{2t}$.

In particular, given that

$$\frac{\ddot{a}}{a} = \dot{H} + H^2, \quad (72)$$

the modified Friedman equations become

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa_5^2 \rho}{4F(R)A(t, y)} - \frac{f(R)}{4F(R)A(t, y)} + \frac{\square F(R)}{4F(R)A(t, y)}, \quad (73)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa_5^2}{8F(R)A(t, y)}(\rho + 3P) - \frac{\dot{A}(t, y)H}{2A(t, y)} + \frac{\square F(R)}{4F(R)A(t, y)} - \frac{f(R)}{8F(R)A(t, y)}. \quad (74)$$

Here we consider the dynamical equations on the brane which is located at $y = 0$. In this case, the EoS parameter reduces to

$$w(t, y = 0) = \frac{4(1 + 16\alpha\lambda)H^2(\frac{1}{4} + \lambda t^2) - 4\lambda(1 + 28\alpha\lambda)}{4(1 + 16\alpha\lambda)H^2(\frac{3}{4} - \lambda t^2) + 4\lambda(1 + 28\alpha\lambda)}. \quad (75)$$

The behavior of the EoS parameter (75) as a function of t is depicted in Figure 2. We immediately see from the figure that, the EoS parameter starts from $w = 1/3$ which is the radiation equation of state parameter and then for large values of t it converges to a constant value $w = -1$ which corresponds to dark energy equation of state parameter. We also plotted the energy density (67) as a function of time on the brane in Figure 3. It is shown that the energy density decreases as the time goes on similar to radiation dominate era where the radiation obeys the standard continuity equation ($\rho_r \propto a(t)^{-4}$), but with a difference that in this model the energy density arrives to a constant value for large t which corresponds to energy density of the dark energy.

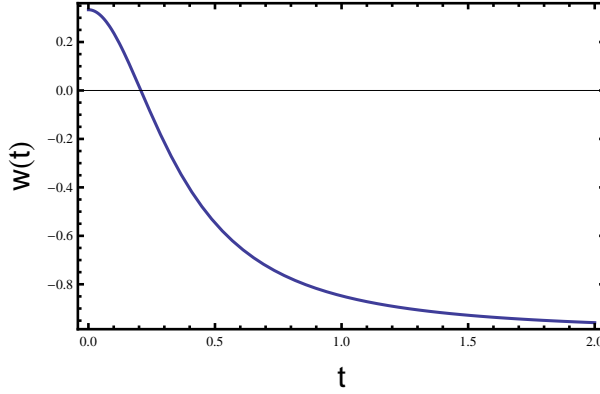


Figure 2: The EoS parameter w as a function of t on the brane($y = 0$) for arbitrary values $\Delta, \alpha = 1$. It starts from $w = 1/3$ to $w = -1$ which corresponds to radiation and dark energy EoS parameter, respectively.

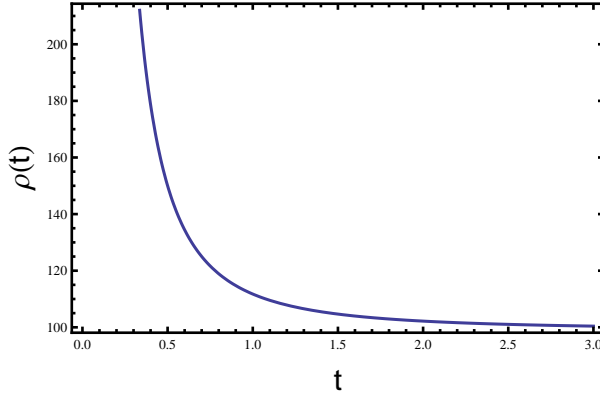


Figure 3: Energy density as a function of time for arbitrary values $\Delta, \alpha = 1$. It is seen that the brane energy density converges to a constant value.

6 Conclusion

In this paper, we have investigated braneworld models with a Gaussian warp function which has a Z_2 symmetry and also with a single extra spatial dimension of infinite extent. We replaced the Einstein-Hilbert Lagrangian by a nonlinear $f(R)$ Lagrangian in vacuum, including an extra dimension. We worked out the $f(R)$ equations of motion, which lead to an exact vacuum $f(R)$ solution. By appropriately setting constants of integration, a bulk cosmological constant can be obtained.

We studied the gravitational fluctuations of our solution by adding small tensor perturbations. We realized that the solution is stable against the perturbations. We also showed that the gravitational zero mode is normalizable and can be localized on the brane. The behavior of the potential was thoroughly addressed, with the minimum of the potential regarded as the stable point which leads to the desired stability.

Finally, we considered the flat FLRW brane model with a Gaussian warp factor and the R^2 approximation. We showed that, the matter which supports the solution starts like a radiation dominated era and in the late time it acts like dark energy with a constant energy density. The equation of state parameter was shown to start from $w = 1/3$ (radiation) and end with $w = -1$ (dark energy).

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